However, during the period of creation of the general theory of relativity Einstein was entirely guided by the equivalence principle in its initial formulation which therefore played an heuristic role in the construction of the theory [32, p. 400]: "The entire theory arose on the basis of the conviction that in a gravitational field all physical processes occur in exactly the same way as without a gravitational field but in an appropriately accelerated (three-dimensional) coordinate system (the "equivalence hypothesis").

Since at that time, thanks to a discovery of G. Minkowski, it was known that to different systems of reference there corresponds a different (and in the general case nondiagonal) metric of space—time, Einstein and Grossman arrived at the conclusion that the field variable for the gravitational field should be the metric tensor of a Riemannian space—time which must be determined by the distribution and motion of matter. There thus arose the idea of the connection of the geometry of space—time with matter.

Proceeding from these considerations, Einstein and Grossman in a purely intuitive manner attempted to establish the form of the equations connecting the components of the metric tensor of Riemannian space—time with the energy—momentum tensor of matter. After long unsuccessful attempts such equations were found by Einstein at the end of 1915.

Since these equations were obtained on the basis of a variational principle somewhat earlier by the mathematician D. Hilbert, we shall call them the Hilbert—Einstein equations.

2. Einstein's Theory of Gravitation

Using the Lagrangian formalism, we shall establish the basic relations of Einstein's theory and also consider a number of questions needed below.

As is known, to find the field equations of any theory it is first necessary to construct a density of the Lagrange function (or simply a Lagrangian density) which should be a scalar density of weight +1. In the general theory of relativity the field variable is the metric tensor of Riemannian space—time g_{ni}; therefore, the simplest Lagrangian density of the gravitational field L_g has the form

$$L_g = \sqrt{-g}R,$$

where g is the determinant of the metric tensor g_{ni}, and R is the scalar curvature of Riemann space—time.

In Einstein's theory the Lagrangian density of matter L_M is usually obtained from the corresponding expression of special relativity written in an arbitrary curvilinear coordinate system by replacing the metric tensor of flat space—time γ_{ni} by the metric tensor of Riemannian space—time g_{ni}. Thus, the action function of the gravitational field and matter in the general gheory of relativity has the form

$$J = -\frac{c^3}{16\pi G}\int \sqrt{-g}R d^4x + \frac{1}{c}\int L_M(\varphi_A, g_{ni}) d^4x, \tag{2.1}$$

where G is the gravitational constant, $G \simeq 6.67 \cdot 10^{-8}$ $cm^3/(g \cdot sec^2)$, c is the velocity of light, and φ_A are the remaining fields of matter.

To obtain the equations of the gravitational field we must vary the action function (2.1) with respect to the components of the metric tensor g_{ni}. Since the expression (2.1) contains covariant as well as contravariant components of the metric tensor, we shall vary the action function with respect to them as independent variables and then consider the relation between their variations

$$\delta g^{nl} = -g^{nl} g^{mi} \delta g_{ml}. \tag{2.2}$$

We can therefore write the expression for the symmetric energy—momentum tensor of matter in Riemannian space—time T^{ni} in the form

$$T^{ni} = -\frac{2}{\sqrt{-g}}\frac{\Delta L_M}{\Delta g_{ni}} = -\frac{2}{\sqrt{-g}}\left[\frac{\delta L_M}{\delta g_{ni}} - g^{nl} g^{im}\frac{\delta L_M}{\delta g^{ml}}\right], \tag{2.3}$$

where $\delta L_M/\delta g_{ni}$ and $\delta L_M/\delta g^{mi}$ are the Euler—Lagrange variations with respect to the covariant and hence the contravariant components of the metric tensor of Riemannian space—time.

By definition, the Euler–Lagrange variation has the form

$$\frac{\delta L}{\delta \varphi}=\frac{\partial L}{\partial \varphi}-\partial_n\left(\frac{\partial L}{\partial(\partial_n \varphi)}\right)+\partial_{nm}\left(\frac{\partial L}{\partial(\partial_{nm}\varphi)}\right)-\ \ldots \tag{2.4}$$

We shall compute the variation δJ for an arbitrary infinitesimal transformation of the field variable g_{ni} without considering either the coordinate system or points of space–time:

$$\delta J=-\frac{c^3}{16\pi G}\int d^4x\{R\delta\sqrt{-g}+\sqrt{-g}R_{ni}\delta g^{ni}+\sqrt{-g}\,g^{ni}\delta R_{ni}\}+\frac{1}{c}\int d^4x\left\{\frac{\delta L_M}{\delta g_{ni}}\,\delta g_{ni}+\frac{\delta L_M}{\delta g^{ml}}\,\delta g^{ml}\right\}.$$

Assuming that on the boundaries of the region of integration of the variation of the metric tensor of Riemannian space–time vanishes and considering the expressions (2.2) and (2.3) and also the relations

$$\delta\sqrt{-g}=\frac{1}{2}\sqrt{-g}\,g^{ni}\delta g_{ni},$$

$$\int d^4x\sqrt{-g}\,g^{ni}\delta R_{ni}=0,$$

from this we have

$$\delta J=\frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\left\{R^{ni}-\frac{1}{2}\,g^{ni}R-\frac{8\pi G}{c^4}T^{ni}\right\}\delta g_{ni}.$$

By the principle of least action, this expression must be equal to zero. Since inside the region of integration $\delta g_{ni} \neq 0$, this condition requires that the Hilbert–Einstein equations be satisfied:

$$R^{ni}-\frac{1}{2}Rg^{nl}=\frac{8\pi G}{c^4}\,T^{ni}. \tag{2.5}$$

In mixed components the Hilbert–Einstein equations (2.5) have the form

$$R_l^{\,n}-\frac{1}{2}\,\delta_l^{\,n}R=\frac{8\pi G}{c^4}\,T_l^{\,n}. \tag{2.6}$$

Contracting indices in Eq. (2.6) and introducing the notation $T = T_n^n$, we have

$$R=-\frac{8\pi G}{c^4}\,T. \tag{2.7}$$

Therefore, the Hilbert–Einstein equations (2.5) can be written in another equivalent form

$$R^{ni}=\frac{8\pi G}{c^4}\left(T^{ni}-\frac{1}{2}\,g^{ni}T\right). \tag{2.8}$$

Away from matter where $T^{ni} = 0$, from expression (2.5) we obtain the vacuum Hilbert–Einstein equations

$$R^{ni}-\frac{1}{2}\,g^{ni}R=0. \tag{2.9}$$

We shall now establish some relations of Einstein's theory needed below.

Since the curvature tensor R^i_{nlm} of Riemannian space–time satisfies the Bianchi–Padov identity

$$\nabla_l R^i_{njm}+\nabla_m R^i_{nlj}+\nabla_j R^i_{nml}=0,$$

raising the index n and contracting the indices ij and nm, we obtain

$$\nabla_n\left(R_l^{\,n}-\frac{1}{2}\,\delta_l^{\,n}R\right)=0.$$

Thus, the covariant divergence in Riemannian space–time of the left side of the Hilbert–Einstein equations (2.6) automatically vanishes. Hence, for self-consistency of Einstein's theory the covariant divergence of the right side of Eqs. (2.6) must also vanish identically.

In order to see this, we consider the second term in the action function (2.1) corresponding to the action function of matter

$$J_M = \frac{1}{c} \int L_M(g_{ni}, \varphi_A) d^4x. \quad (2.10)$$

Since the first term in expression (2.1) does not depend on the fields of matter φ_A, varying the action function of matter (2.10) with respect to the fields φ_A, we obtain the equations of motion of matter in Riemannian space–time

$$\frac{\delta L_M}{\delta \varphi_A} = 0. \quad (2.11)$$

We now make an infinitesimal transformation of the coordinate system

$$x'^i = x^i + \xi^i(x), \quad (2.12)$$

where ξ^i is an infinitely small four-vector.

This transformation generates coordinate variation of the functions $\delta_c \varphi_A$ and $\delta_c g_{ni}$ and also the limits of integration in the expression (2.10). Up to linear terms in the infinitely small quantity $\xi^i(x)$ we have

$$\delta_c \varphi_A(x) = \varphi'_A(x') - \varphi_A(x) = \varphi'_A(x) - \varphi_A(x) + \frac{\partial \varphi_A}{\partial x^m} \xi^m(x). \quad (2.13)$$

From this it follows that coordinate variation does not commute with the operation of partial differentiation:

$$\delta_c \frac{\partial \varphi_A(x)}{\partial x^m} = \frac{\partial}{\partial x^m} \delta_c \varphi_A(x) - \frac{\partial \varphi_A(x)}{\partial x^n} \frac{\partial \xi^n(x)}{\partial x^m}.$$

Noncommutativity of coordinate variation with partial differentiation is connected with the fact that in definition (2.13) the difference of field functions before and after the coordinate transformation (2.12) is taken, while the argument of the field function after transformation is the image of the argument of the field function before variation, i.e., formally they have different arguments, although they pertain to the same point of space–time. Coordinate variation also does not commute with the operation of integration:

$$\delta_c \int d^4x f(x) = \int d^4x \left\{ \delta_c f(x) + f(x) \frac{\partial \xi^n(x)}{\partial x^n} \right\}.$$

We note also that coordinate variation of a scalar is equal to zero:

$$\delta_c \psi(x) = \psi'(x') - \psi(x) = 0.$$

With the view of application below of coordinate transformation (2.12) to the action function (2.10), it is useful to separate out from coordinate variation the variation in the spirit of the Lie differential which commutes with partial differentiation.

We have by definition

$$\delta_L \varphi_A(x) = \varphi'_A(x) - \varphi_A(x). \quad (2.14)$$

The coordinate variation can then be written in the form

$$\delta_c \varphi_A(x) = \varphi'_A(x') - \varphi'_A(x) + \delta_L \varphi_A(x).$$

Retaining only terms linear in $\xi^n(x)$, we obtain finally

$$\delta_c \varphi_A(x) = \delta_L \varphi_A(x) + \frac{\partial \varphi_A(x)}{\partial x^n} \xi^n(x).$$

Symbolically this result can be represented in the form

$$\delta_c = \delta_L + \xi^n(x) \frac{\partial}{\partial x^n},$$

whereby

$$\delta_L \frac{\partial}{\partial x^n} = \frac{\partial}{\partial x^n} \delta_L.$$

Since the action function of matter (2.10) is a scalar, under coordinate transformation (2.12) it does not change: $\delta_C J_M = 0$. This implies that

$$\int d^4x \{\delta_L L_M + \nabla_n (L_M \xi^n)\} = 0.$$

Taking the Lie variation of the Lagrangian density of matter, we obtain

$$\int d^4x \left\{\frac{\Delta L_M}{\Delta g_{ni}} \delta_L g_{ni} + \frac{\delta L_M}{\delta \varphi_A} \delta_L \varphi_A + \nabla_n J^n\right\} = 0,$$

where the explicit form of the current J^n is inconsequential.

If the equations of motion of matter (2.11) are satisfied, this expression simplifies:

$$\int d^4x \left\{\frac{\Delta L_M}{\Delta g_{ni}} \delta_L g_{ni} + \nabla_n J^n\right\} = 0. \tag{2.15}$$

In relation (2.15) the Lie variation of each of the 10 components of the metric tensor of Riemannian space—time are not independent and can be expressed in terms of the four components of the vector ξ^n. We shall find this dependence. By definition (2.14) we have

$$\delta_L g^{ni} = g'^{ni}(x) - g^{ni}(x). \tag{2.16}$$

Since under coordinate transformations the metric tensor of Riemannian space—time possesses the transformation law

$$g'^{ml}(x') = \frac{\partial x'^m}{\partial x^n} \frac{\partial x'^l}{\partial x^i} g^{ni}(x(x')),$$

in the case of transformation (2.12) up to linear terms in ξ^n we obtain

$$g'^{ml}(x+\xi) = g^{lm}(x) + g^{ls}(x)\,\partial_s \xi^m + g^{ms}(x)\,\partial_s \xi^l.$$

From this it follows that

$$g'^{ml}(x) = g^{ml}(x) + g^{ls}(x)\,\partial_s \xi^m + g^{ms}(x)\,\partial_s \xi^l - \xi^s(x)\,\partial_s g^{ml}(x).$$

Substituting this relation into expression (2.16), we have

$$\delta_L g^{ml}(x) = g^{nl}(x)\,\partial_n \xi^m + g^{mn}(x)\,\partial_n \xi^l(x) - \xi^n(x)\,\partial_n g^{lm}(x) = \nabla^l \xi^m + \nabla^m \xi^l.$$

Because of the relation

$$\delta_L g_{ni} = -g_{nm} g_{il} \delta_L g^{ml},$$

we obtain finally

$$\delta_L g_{ni} = -\nabla_n \xi_i - \nabla_i \xi_n.$$

Considering this expression, we transform relation (2.15) to the form

$$\int d^4x \left\{2\xi_n \nabla_i \frac{\Delta L_M}{\Delta g_{ni}} + \nabla_n \left[J^n - 2\xi_i \frac{\Delta L_M}{\Delta g_{ni}}\right]\right\} = 0.$$

Since the components of the vectors ξ^n inside the region of integration and on its surface are independent and arbitrary, this implies that

$$-2\nabla_i \frac{\Delta L_M}{\Delta g_{ni}} = \nabla_i [\sqrt{-g} T^{ni}] = \sqrt{-g} \nabla_i T^{ni} = 0. \tag{2.17}$$

Thus, the covariant divergence in Riemannian space—time of the right side of the Hilbert—Einstein equations is identically zero.

One of the first exact solutions of the Hilbert—Einstein equations found in the general theory of relativity is the Schwarzschild solution describing a static, spherically symmetric gravitational field. Such a field can be created by any source the matter in which is distributed in a spherically symmetric manner. Spherical symmetry of the gravitational field means that the metric of Riemannian space time in the present case must be identical at all points an equal distance from the source.

Proceeding from the symmetry of the problem, we shall determine which components of the metric tensor of Riemann space—time will be nonzero in this case. We place the origin of the coordinate system at the center of the source. Under rotations of this coordinate system by an arbitrary angle the physical situation must not change due to the spherical symmetry of the distribution of matter. Therefore, the components of the metric tensor g_{ni} after rotation must be the same functions of the transformed argument as the original functions of their original arguments, i.e., their tensor must be form-invariant under rotation of the coordinate system.

This implies that in spherical coordinates only the components $g_{ni}=\{g_{00}(r', t'),\ g_{0r}(r', t'),\ g_{\varphi\varphi}(r', t')=g_{\theta\theta}(r', t')\sin^2\theta\}$, can be nonzero components of the metric tensor of Riemannian space—time, since only in this fashion is the tensor g_{ni} form-invariant under rotation. However, the condition of spherical symmetry still does not establish the final form of the metric tensor of Riemannian space—time, since we still have the possibility of making any admissible* transformations of coordinates and time $r' = r'(r, t)$ and $t' = t'(r, t)$ which do not destroy the condition of spherical symmetry of the metric but alter the magnitude of the components g_{00}, g_{0r} and g_{rr} of the metric tensor g_{ni}.

Usually the functions $r' = r'(r, t)$, $t' = t'(r, t)$ are chosen so that the component g_{0r} in the new coordinate system vanishes, while the component $g_{\theta\theta}$ coincides with its pseudo-Euclidean limit $g_{\theta\theta} = -r^2$. Coordinates satisfying these conditions are called Schwarzschild coordinates. A distinguishing feature of these coordinates is that the length of a circle with center at the origin is equal to $2\pi r$. However, for our purposes it is more convenient to use isotropic spherical coordinates in which the spatial part of an interval is conformally Euclidean.

We therefore subject the choice of functions $r' = r'(t, r)$ and $t' = t'(r, t)$ to the conditions

$$g_{0r}=0,\ g_{\theta\theta}=r^2 g_{rr}.$$

The static, spherically symmetric gravitational field in isotropic coordinates will then be described by the metric

$$ds^2=g_{00}(r)\,c^2dt^2+g_{rr}(r)\,[dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)]. \tag{2.18}$$

Since in admissible coordinates the conditions $g_{00} > 0$, $g_{rr} < 0$ must be satisfied, it is convenient to write expression (2.18) in the form

$$ds^2=c^2e^{2\nu}dt^2-e^{2\lambda}[dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)], \tag{2.19}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$ are unknown functions which must be determined from the Hilbert—Einstein equations.

Thus, the convariant components of the metric tensor of Riemannian space—time in the present case have the form

$$\begin{gathered} g_{00}=e^{2\nu};\quad g_{rr}=-e^{2\lambda};\\ g_{\theta\theta}=-r^2e^{2\lambda};\quad g_{\varphi\varphi}=-r^2\sin^2\theta e^{2\lambda}. \end{gathered} \tag{2.20}$$

For the contravariant components of the metric tensor we have

$$\begin{gathered} g^{00}=e^{2-\nu};\quad g^{rr}=-e^{-2\lambda};\\ g^{\theta\theta}=-\frac{1}{r^2}e^{-2\lambda};\quad g^{\varphi\varphi}=-\frac{1}{r^2\sin^2\theta}e^{-2\lambda}. \end{gathered} \tag{2.21}$$

Using expressions (2.20) and (2.21), it is easy to determine the nonzero components of the connection of Riemannian space—time:

$$\Gamma^0_{0r}=\nu';\quad \Gamma^r_{00}=\nu' e^{2(\nu-\lambda)};\quad \Gamma^r_{rr}=\lambda';$$

*Transformations of coordinates of systems of reference which can be realized by real physical bodies and processes we call admissible transformations. Mathematically, this condition is equivalent to the requirement [28] that in these reference systems the quadratic form with coefficients $g_{\alpha\beta}$ be negative definite, while the component g_{00} of the metric tensor is positive definite: $g_{00} > 0$, $g_{\alpha\beta}dx^\alpha dx^\beta < 0$.

$$\Gamma^r_{\theta\theta} = -(r + r^2\lambda'); \quad \Gamma^r_{\varphi\varphi} = -(r + r^2\lambda')\sin^2\theta;$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta; \quad \Gamma^\varphi_{\theta\varphi} = \frac{\cos\theta}{\sin\theta}; \tag{2.22}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r} + \lambda',$$

where the prime denotes the derivative with respect to r. Computing the components of the Ricci tensor and substituting them into the Hilbert–Einstein equations (2.16), we obtain

$$\begin{aligned} -\left[2\lambda'' + \frac{4}{r}\lambda' + \lambda'^2\right]e^{-2\lambda} &= \frac{8\pi G}{c^4}T_0{}^0;\\ -\left[\lambda'^2 + 2\lambda'\nu' + \frac{2}{r}\lambda' + \frac{2}{r}\nu'\right]e^{-2\lambda} &= \frac{8\pi G}{c^4}T_r{}^r;\\ -\left[\lambda'' + \nu'' + \frac{1}{r}\lambda' + \frac{1}{r}\nu' + \nu'^2\right]e^{-2\lambda} &= \frac{8\pi G}{c^4}T^\theta_\theta;\\ -\left[\lambda'' + \nu'' + \frac{1}{r}\lambda' + \frac{1}{r}\nu' + \nu'^2\right]e^{-2\lambda} &= \frac{8\pi G}{c^4}T^\varphi_\varphi. \end{aligned} \tag{2.23}$$

To find the metric of Riemannian space–time outside a static, spherically symmetric source we set the right sides of Eqs. (2.23) equal to zero. Then the two functions $\lambda(r)$ and $\nu(r)$ must satisfy the three equations:

$$\begin{aligned} 2\lambda'' + \frac{4}{r}\lambda' + \lambda'^2 &= 0,\\ \lambda'^2 + 2\lambda'\nu' + \frac{2}{r}\lambda' + \frac{2}{r}\nu' &= 0,\\ \lambda'' + \nu'' + \frac{1}{r}\lambda' + \frac{1}{r}\nu' + \nu'^2 &= 0. \end{aligned} \tag{2.24}$$

It is easy to see that the first equation of this system can be written in the form

$$2(r^2\lambda')' + \frac{1}{r^2}(r^2\lambda')^2 = 0.$$

Integrating this ordinary differential equation on $r^2\lambda'$, we have

$$r^2\lambda' = -\frac{2C_1 r}{r + C_1},$$

where C_1 is a constant of integration. Hence, it follows that

$$\lambda = C_2 + 2\ln\left[1 + \frac{C_1}{r}\right].$$

Since as $r \to \infty$ the components $g_{\alpha\beta}$ of the metric tensor of Riemannian space–time must satisfy the condition

$$\lim_{r\to\infty} g_{\alpha\beta} = \lim_{r\to\infty} \gamma_{\alpha\beta}e^{2\lambda} = \gamma_{\alpha\beta},$$

the constant of integration C_2 must be set equal to zero:

$$\lambda = 2\ln\left[1 + \frac{C_1}{r}\right]. \tag{2.25}$$

Substituting expression (2.25) into the second equation of the system (2.24), we obtain

$$\nu'_1 = \frac{2C_1}{r^2 - C_1^2}.$$

Hence, the function $\nu(r)$ has the form

$$\nu = C_3 + \ln\frac{r - C_1}{r + C_1}. \tag{2.26}$$

Using expressions (2.25) and (2.26), it is easy to see that the third equation of system (2.24) is satisfied identically.

We now determine the constants of integration C_1 and C_3. Substituting expression (2.26) into relation (2.20), we obtain

$$g_{00}=e^{2C_3}\cdot\left[\frac{r-C_1}{r+C_1}\right]^2.$$

Since as $r \to \infty$ the component g_{00} of the metric tensor of Riemannian space-time should have the Galilean value $g_{00} = 1$, the constant C_3 should be set equal to zero. Expression (2.26) then assumes the form

$$\nu=\ln\frac{r-C_1}{r+C_1}. \tag{2.27}$$

The constant of integration C_1, on the other hand, by the Hilbert–Einstein equations (2.23) can be expressed as an integral over the volume of the source of the components of the energy-momentum tensor of matter. For this we multiply each equation of (2.23) by $(1/2)\times e^{\nu+3\lambda}$ and subtract from the first equation the remaining equations. As a result, we obtain the following relation:

$$e^{\nu+\lambda}\left[\nu''+\nu'^2+\frac{2}{r}\nu'+\lambda'\nu'\right]=\frac{4\pi G}{c^4}[T_0^0-T_r^r-T_\theta^\theta-T_\varphi^\varphi]\,e^{\nu+3\lambda}.$$

After the identity transformation, we have

$$[r^2\nu' e^{\nu+\lambda}]'=\frac{4\pi G}{c^4}[T_0^0-T_r^r-T_\theta^\theta-T_\varphi^\varphi]\,e^{\nu+3\lambda}\cdot r^2.$$

The right side of this equality is nonzero for $0 < r < a$, where a is the radius of the source in isotropic coordinates. Therefore, integrating the given equation on r in the interval from zero to $r > a$ and assuming that for the interior solution the functions $\nu(r)$ and $\lambda(r)$ have no singularities, we obtain

$$r^2\nu' e^{\nu+\lambda}=\frac{4\pi G}{c^4}\int_0^a r^2 dr\,[T_0^0-T_r^r-T_\theta^\theta-T_\varphi^\varphi]\exp(\nu+3\lambda).$$

Since away from the source the functions $\nu(r)$ and $\lambda(r)$ are defined by the expressions (2.25) and (2.27), the left side of this relation assumes the form

$$2C_1=\frac{4\pi G}{c^4}\int_0^a r^2 dr\,[T_0^0-T_r^r-T_\theta^\theta-T_\varphi^\varphi]\exp(\nu+3\lambda).$$

Using the Hilbert–Einstein equations and noting that in the spherically symmetric case $4\pi r^2 \exp(\nu + 3\lambda)$ corresponds to $\sqrt{-g}dV$, we obtain finally

$$C_1=\frac{G}{c^4}\int\sqrt{-g}dV\left[T_0^0-\frac{1}{2}T\right]=\frac{1}{8\pi}\int\sqrt{-g}R_0^0 dV.$$

On the other hand, since as $r \to \infty$ it is necessary to ensure that Newton's law of gravitation holds, the constant of integration C_1 must be connected with the gravitational mass M of the source by the relation

$$C_1=\frac{GM}{2c^2}.$$

From this it follows that in Einstein's theory the gravitational mass of a static, spherically symmetric source is determined by the expression

$$M=\frac{c^2}{4\pi G}\int\sqrt{-g}R_0^0 dV=\frac{1}{c^2}\int\sqrt{-g}dV\,[T_0^0-T_r^r-T_\theta^\theta-T_\varphi^\varphi]. \tag{2.28}$$

Thus, outside a static, spherically symmetric source the metric has the form

$$ds^2=\left[\frac{1-\frac{r_g}{4r}}{1+\frac{r_g}{4r}}\right]^2 c^2dt^2-\left[1+\frac{r_g}{4r}\right]^4[dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)], \tag{2.29}$$

where $r_g = 2GM/c^2$ is the gravitational radius of the source.

As Birkhoff showed, this metric describes also the gravitational field outside a nonstatic source with a distribution and notion of matter which is spherically symmetric.

Introducing isotropic Cartesian coordinates x, y, z in correspondence with the equalities

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta,$$

the interval (2.29) can be written in the form

$$ds^2=\left[\frac{1-\frac{r_g}{4r}}{1+\frac{r_g}{4r}}\right]^2 c^2dt^2-\left[1+\frac{r_g}{4r}\right]^4[dx^2+dy^2+dz^2], \tag{2.30}$$

where we have introduced the notation $r = \sqrt{x^2 + y^2 + z^2}$.

In Einstein's theory the Schwarzschild metric is very often used as a touchstone in resolving many questions. The broad application of this metric is explained by the fact that it represents an exact solution of the Hilbert–Einstein equations and is a sufficiently good idealization for describing the gravitational field outside real astrophysical objects for which the deviations of the distribution and the motion of matter from spherical symmetry are small.

Due to the essential nonlinearity of the Hilbert–Einstein equations, in the majority of practically important cases it is not possible to find exact solutions of these equations. In particular, at the present time in the theory of general relativity no exact solution describing the metric outside an insular source of gravitational radiation has been found in explicit form.

Therefore, together with the search for exact solutions, analysis of various questions in Einstein's theory are carried out on the basis of approximate Hilbert–Einstein equations. One such case is the study of the propagation of small wavelike perturbations of the metric tensor of Riemannian space–time. Leaving aside the question of the legitimacy of applying the method of successive approximations to the Hilbert–Einstein equations (see, for example, [15]), we shall show how such an analysis is usually carried out in the general theory of relativity.

To first approximation in the coupling constant it is usually assumed that the source of gravitational radiation creates weak gravitational perturbations on the background of flat space–time. The metric tensor of Riemannian space–time in this case is written in the form

$$g_{ni}=\gamma_{ni}+\frac{G}{c^4}\varphi_{ni}+\dots, \tag{2.31}$$

where γ_{ni} is the Galilean metric tensor with signature (+, –, –, –). Substituting the expression (2.31) into the Hilbert–Einstein equation (2.15) and linearizing them in G/c^4, we obtain

$$\Box\psi_{nm}-\partial_n\partial^i\psi_{mi}-\partial_m\partial^i\psi_{ni}+\gamma_{nm}\partial_i\partial_l\psi^{li}=-16\pi T_{nm}, \tag{2.32}$$

where $\Box=\partial_i\partial^i$, $\psi_{nm}=\varphi_{nm}-\frac{1}{2}\gamma_{nm}\varphi_i^{\,i}$ and all operations of lifting and lowering tensor indices are carried out by means of the metric tensor γ_{nm}. Since the energy–momentum tensor of matter in Eq. (2.32) must be taken in zeroth approximation in G/c^4, from the covariation equation (2.17) it follows that it satisfies the relation

$$\partial^n T_{nm}=0. \tag{2.33}$$

Equations (2.32) in the approximation considered are invariant under transformations of the tensor ψ_{nm}:

$$\psi_{nm}\to\psi_{nm}-\partial_n\xi_m-\partial_m\xi_n+\gamma_{nm}\partial_l\xi^l, \tag{2.34}$$

where ξ_l is an arbitrary four-vector. We can use this arbitrariness in the choice of the tensor ψ_{nm} to ensure that the additional Hilbert–de Donder equations are satisfied:

$$\partial_n\psi^{nm}=0. \tag{2.35}$$

Equations (2.32) then take the form

$$\Box\psi_{nm} = -16\pi T_{nm}. \tag{2.36}$$

We write the tensors ψ_{nm} and T_{nm} as Fourier integrals in the time:

$$\psi_{nm}(\vec{r}, t) = \int e^{-i\omega t}\tilde{\psi}_{nm}(\vec{r}, \omega)\, d\omega,$$

$$T_{nm}(\vec{r}, t) = \int e^{-i\omega t}\tilde{T}_{nm}(\vec{r}, \omega)\, d\omega.$$

In the spectrum $\tilde{T}_{nm}$ we distinguish the static part $\tilde{T}_{nm}(0)$. It is obvious that the static part of the tensorial current will give only static solutions; we therefore omit it. Equations (2.36) then assume the form

$$\Delta\tilde{\psi}_{nm} + \omega^2\tilde{\psi}_{nm} = 16\pi\tilde{T}_{nm}.$$

A solution of these equations can be written in the form ($R = |\vec{r} - \vec{r}'|$);

$$\tilde{\psi}_{nm} = -4\int \frac{\exp(i\omega R)}{R}\tilde{T}_{nm}(\vec{r}', \omega)\, dV.$$

Using the Hilbert–de Donder conditions (2.35) $i\omega\tilde{\psi}^{0n} = \partial_\alpha\tilde{\psi}^{\alpha n}$, we express the components $\tilde{\psi}^{0n}$ in terms of the spatial components:

$$\tilde{\psi}^{00} = -\frac{1}{\omega^2}\partial_\alpha\partial_\beta\tilde{\psi}^{\alpha\beta},$$

$$\tilde{\psi}^{0\alpha} = -\frac{i}{\omega}\partial_\beta\tilde{\psi}^{\alpha\beta}$$

Outside the source of gravitational waves by a transformation (2.34) consistent with the Hilbert–de Donder conditions (2.35) for $\Box\xi^n = 0$, we can impose on the wave components of the metric ψ_{nm} four more conditions according to the number of independent vectors ξ^n. For such conditions it is possible to choose the following: $\tilde{\psi}'^{0\alpha} = 0$, $\tilde{\psi}'^{n}_{n} = 0$ (TT gauge).

As a result of this transformation, we obtain

$$\tilde{\psi}'^{\alpha\beta} = \tilde{\psi}^{\alpha\beta} - \frac{1}{2}\gamma^{\alpha\beta}\tilde{\psi}_n{}^n - \frac{i}{\omega}[\partial^\beta\tilde{\psi}^{0\alpha} + \partial^\alpha\tilde{\psi}^{0\beta}] - \frac{1}{\omega^2}\partial^\alpha\partial^\beta\left[\tilde{\psi}^{00} - \frac{1}{2}\tilde{\psi}_n{}^n\right],$$

$$\tilde{\psi}'^{0\alpha} = 0, \quad \tilde{\psi}'^{n}_{n} = 0.$$

Considering the Hilbert–de Donder conditions (2.35), we can write these expressions in the form

$$\tilde{\psi}'^{\alpha\beta} = \tilde{S}^{\alpha\beta} - \frac{1}{\omega^2}[\partial^\beta\partial_\nu\tilde{S}^{\nu\alpha} + \partial^\alpha\partial_\nu\tilde{S}^{\nu\beta}] + \frac{1}{2\omega^2}\gamma^{\alpha\beta}\partial_\nu\partial_\mu\tilde{S}^{\mu\nu} + \frac{1}{2\omega^4}\partial^\alpha\partial^\beta\partial_\nu\partial_\mu\tilde{S}^{\mu\nu}, \tag{2.37}$$

where we have introduced the notation

$$\tilde{S}^{\alpha\beta} = \psi^{\alpha\beta} - \frac{1}{3}\gamma^{\alpha\beta}\tilde{\psi}_n{}^n. \tag{2.38}$$

Thus, in first order of perturbation theory the wave solution of the Hilbert–Einstein equations contains in the general case six nonzero spatial components $\tilde{\psi}'^{\alpha\beta}$, but only two of these are independent, because of the four Hilbert–de Donder conditions (2.35) and the equality $\psi'^{n}_{n} = 0$ expressing that the trace is zero. These additional conditions represent the known additional conditions for an irreducible representation with spin 2 in the TT gauge; hence, in Einstein's theory the perturbation of the metric to first order in the constant G/c^4 has spin 2.

Usually the wave solutions of Eqs. (2.36) are written in a somewhat different form which makes it possible to graphically demonstrate the quadrupole character of linear perturbations of the metric in Einstein's theory. For this we note that the spatial components $\tilde{\psi}^{\alpha\beta}$ by Eq. (2.33) can be written in the form

$$\tilde{\psi}^{\alpha\beta} = -4\int \frac{\exp(i\omega R)}{R}\tilde{T}^{\alpha\beta}dV = 2\omega^2\left\{\int \frac{\exp(i\omega R)}{R}\tilde{T}^{00}x^\alpha x^\beta dV \right.$$
$$\left. + \frac{2i}{\omega}\partial_\nu\int \frac{\exp(i\omega R)}{R}\tilde{T}^{0\nu}x^\alpha x^\beta dV - \frac{1}{\omega^2}\partial_\nu\partial_\mu\int \frac{\exp(i\omega R)}{R}\tilde{T}^{\nu\mu}x^\alpha x^\beta dV\right\}. \tag{2.39}$$

This relation is exact. It simplifies considerably if the linear dimensions of the source are considerably less than the distance from its center to the point of observation. Omitting nonwave terms decreasing faster than 1/r, we obtain

$$\tilde{\psi}^{\alpha\beta}=\frac{2\omega^2}{r}\int dV x^\alpha x^\beta \exp(i\omega R)[\tilde{T}^{00}+2n_\nu\tilde{T}^{0\nu}+n_\nu n_\mu\tilde{T}^{\nu\mu}],$$

where $n^\nu = x^\nu/r$, $n_\nu n^\nu = -1$. Expression (2.38) can then be written in the form

$$\tilde{S}^{\alpha\beta}=\frac{2\omega^2}{r}\int dV\left[x^\alpha x^\beta-\frac{1}{3}\gamma^{\alpha\beta}x_\nu x^\nu\right][\tilde{T}^{00}+2n_\mu\tilde{T}^{0\mu}+n_\mu n_\eta\tilde{T}^{\mu\eta}]\exp(i\omega R). \quad (2.40)$$

Introducing the projection operators

$$P^\alpha_\beta=\delta^\alpha_\beta+n^\alpha n_\beta, \quad (2.41)$$

satisfying the conditions

$$P^\alpha_\alpha=2, \quad P^\alpha_\beta P^\beta_\nu=P^\alpha_\nu,$$

we can rewrite relation (2.37) in the form

$$\tilde{\psi}'^{\alpha\beta}=\left[P^\alpha_\nu P^\beta_\mu-\frac{1}{2}P^{\alpha\beta}P_{\nu\mu}\right]\tilde{S}^{\mu\nu}. \quad (2.42)$$

Substituting expression (2.40) into the Fourier integral, we obtain

$$S^{\alpha\beta}=-\frac{2}{r}\frac{d^2}{dt^2}\int dV\left(x^\alpha x^\beta-\frac{1}{3}\gamma^{\alpha\beta}x_\nu x^\nu\right)[T^{00}+2n_\delta T^{0\delta}+n_\delta n_\mu T^{\delta\mu}]\,\mathrm{ret}.$$

Here [...]ret denotes that the expression in square brackets is taken at the retarded time $t' = t - R/c$. If we introduce the traceless tensor of the generalized quadrupole moment

$$\mathscr{D}^{\alpha\beta}=D^{\alpha\beta}+2n_\nu D^{\alpha\beta\nu}+n_\nu n_\mu D^{\alpha\beta\nu\mu},$$

where

$$D^{\alpha\beta}=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\nu x^\nu)[T^{00}]\,\mathrm{ret},$$

$$D^{\alpha\beta\nu}=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\delta x^\delta)[T^{0\nu}]\,\mathrm{ret},$$

$$D^{\alpha\beta\mu\nu}=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\delta x^\delta)[T^{\mu\nu}]\,\mathrm{ret},$$

then the perturbation components of metric (2.31) can be written in the form

$$\varphi^{\alpha\beta}=h^{\alpha\beta}/r; \quad \varphi^{0i}=0, \quad \varphi_n{}^n=0,$$
$$h^{\alpha\beta}=-\frac{2}{3}\left(P^\alpha_\nu P^\beta_\mu-\frac{1}{2}P^{\alpha\beta}P_{\mu\nu}\right)\ddot{\mathscr{D}}^{\mu\nu}. \quad (2.43)$$

Here and below the dot denotes the derivative with respect to time.

Since usually rather slow motions of the source are considered for which there are the estimates

$$|\ddot{T}^{00}|\gg|\ddot{T}^{0\alpha}|\gg|\ddot{T}^{\alpha\beta}|,$$

in expression (2.43) it is possible to set

$$\mathscr{D}^{\alpha\beta}=\int dV\,(3x^\alpha x^\beta-\gamma^{\alpha\beta}x_\delta x^\delta)\,T^{00}\left(t-\frac{r}{c}\right).$$

From expressions (2.31) and (2.43) it is easily found that the nonzero components of the curvature tensor in first approximation have the form

$$R_{0\alpha0\beta}=-\frac{G}{2c^6r}\ddot{h}_{\alpha\beta}\left[1+O\left(\frac{Gh_{ik}}{c^4r}\right)\right],$$
$$R_{\alpha\beta0\nu}=\frac{G}{2c^6r}[\ddot{h}_{\alpha\nu}n_\beta-\ddot{h}_{\beta\nu}n_\alpha]\left[1+O\left(\frac{Gh_{ni}}{c^4r}\right)\right],$$

$$R_{\alpha\beta\mu\nu}=\frac{G}{2c^6r}\left[\ddot{h}_{\alpha\nu}n_\beta n_\mu+\ddot{h}_{\beta\mu}n_\alpha n_\nu-\ddot{h}_{\alpha\mu}n_\beta n_\nu-\ddot{h}_{\beta\nu}n_\alpha n_\mu\right]\left[1+O\left(\frac{Gh_{ni}}{c^4r}\right)\right]. \tag{2.44}$$

In a region of space with linear dimensions considerably less than the distance from its center to the source of the metric waves, the spherical wave (2.43) may be considered an elliptically polarized wave. Assuming, to be specific, that the region in question and the source of the metric waves are located on the z axis ($n^3 = 1$, $n_1 = n^2 = 0$), for the nonzero components we obtain

$$\begin{gathered}\varphi^{11}=-\varphi^{22}=-\frac{1}{3r_0}[\ddot{D}^{11}-\ddot{D}^{22}],\\ \varphi^{12}=-\frac{2}{3r_0}\ddot{D}^{12},\end{gathered} \tag{2.45}$$

where r_0 is the distance from the source to the center of the region in question. In the case of radiation of monochromatic waves, the second derivatives with respect to time of the components of the quadrupole moment can be written in the form

$$\begin{gathered}\ddot{D}^{11}=d^{11}\exp[i(kz-\omega t)],\\ \ddot{D}^{22}=d^{22}\exp[i(kz-\omega t)],\\ \ddot{D}^{12}=id^{12}\exp[i(kz-\omega t)].\end{gathered}$$

Therefore, from expression (2.45) we have

$$\begin{gathered}\varphi^{11}=-\varphi^{22}=h_0\cos 2\psi\exp[i(kz-\omega t)],\\ \varphi^{12}=ih_0\sin 2\psi\exp[i(kz-\omega t)],\end{gathered} \tag{2.46}$$

where $h_0=\frac{1}{3r_0}\sqrt{4(d^{12})^2+(d^{11}-d^{22})^2}$, $\operatorname{tg} 2\psi=\frac{2d^{12}}{d^{11}-d^{22}}$.

The degree of ellipticity of the polarization of the gravitational wave (2.46) is measured by the quantity $\tan 2\psi$. If $\tan 2\psi = 0$ or ∞, then the wave is linearly polarized; if $|\tan 2\psi| = 1$, then the wave is circularly polarized. For other values of $\tan 2\psi$ the wave is elliptically polarized. If $\tan 2\psi > 0$, then the wave has right-hand polarization, while if $\tan 2\psi < 0$ it has left-hand polarization.

For the nonzero components of the curvature wave in the case we consider we have

$$\begin{gathered}R_{0101}=R_{0113}=R_{1313}=-R_{2323}=R_{0223}=-R_{0202}=-\frac{G}{2c^6}\ddot{\varphi}^{11},\\ R_{0102}=R_{1323}=R_{0213}=-\frac{G}{2c^6}\ddot{\varphi}^{12}.\end{gathered} \tag{2.47}$$

It should be noted that the magnitude of the components of the curvature tensor (2.47) does not depend on the dimensions of the region considered.

3. Energy–Momentum Pseudotensors of the Gravitational Field in the General Theory of Relativity

Einstein considered that in the general theory of relativity the gravitational field with matter should possess some conservation law [32, p. 299]: "... by all means it should be required that matter and the gravitational field together satisfy laws of conservation of energy–momentum." In the opinion of Einstein this problem was completely resolved on the basis of "conservation laws" using the energy–momentum pseudotensor as the energy–momentum characteristic of the gravitational field.

To obtain such "conservation laws" one usually [11] proceeds as follows. If the Hilbert–Einstein equations are written in the form

$$-\frac{c^4}{8\pi G}g\left[R^{ik}-\frac{1}{2}g^{ik}R\right]=-gT^{ik}, \tag{3.1}$$

then the left side can be identically represented as the sum of two noncovariant quantities

$$-\frac{c^4}{8\pi G}\left[R^{ik}-\frac{1}{2}g^{ik}R\right]g=\frac{\partial}{\partial x^l}h^{ikl}+g\tau^{ik}, \tag{3.2}$$